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Kinematic segregation of granular mixtures in sandpiles

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Abstract. We study the segregation of granular mixtures in two-dimensional silos using a recently proposed set of coupled equations for surface flows of grains. We study the thick flow regime, where the grains are segregated in the rolling phase. We incorporate this dynamical segregation process, called kinematic sieving, free-surface segregation or percolation, into the theoretical formalism and calculate the profiles of the rolling species and the concentration of grains in the bulk in the steady state. Our solution shows the segregation of the mixture with the large grains being found at the bottom of the pile in qualitative agreement with experiments.

PACS. 83.70.Fn Granular solids – 83.10.Pp Particle dynamics – 47.55.Kf Multiphase and particle-laden flows

1 Introduction

When a mixture of grains [1–6] differing in size and shape is poured in a two-dimensional cell consisting of two parallel vertical slabs separated by a narrow gap (*i.e.*, a granular Hele-Shaw cell) the grains segregate according to their size and shape. The typical experimental set consists of a vertical "quasi-two-dimensional" Hele-Shaw cell with a narrow gap of 5 mm separating two transparent plates of 300 mm by 200 mm (see Fig. 1). We close the left edge of the cell leaving the right edge free, and we pour continuously, near the left edge, an equal-volume mixture of grains differing in size and shape. When the larger grains are more rounded than the small grains, a segregation of the mixture is observed with the large grains being found at the bottom of the cell and the small at the top. When the large grains are rougher than the small grains a periodic pattern consisting of layers of large and small grains parallel to the pile surface occurs [7–13]. This periodic pattern is called spontaneous stratification and it was found to occur for a wide range of size ratios between the grains: $d_2/d_1 > 1.4$, where d_1 is the typical size of the small grains and d_2 is the typical size of the large grains. When the size ratio between the grains is close to one, only segregation is found no matter the shape of the grains, *i.e.* no matter the angle of repose of the species.

An important segregation mechanism acting when d_2/d_1 is not close to one, and in the the case of thick flows is the so-called kinematic sieving [14], free surface segregation [15] or percolation effect [8,10]. Due to this phenomenon the large grains in the rolling phase are found to rise to the top of the rolling phase while the small grains sink downward through the gaps left by the motion of larger grains in the rolling phase (Fig. 1). Since only the small grains interact with the surface, they are captured

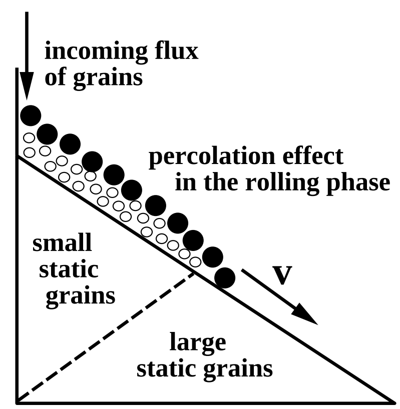

Fig. 1. Schematic representation of the percolation effect in the rolling phase. We pour continuously a mixture of large and small grain in a two-dimensional cell and a steady thick flow of grains is observed. The percolation effect consists on the size segregation of grains in the rolling phase: large grains are observed to rise to the top of the rolling phase, and small grains drifts to the bottom of the rolling phase. Due to the percolation effect, the small grains are the first to be captured at the pile surface resulting in the segregation of the mixture in the bulk.

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at the surface first causing the larger grains to be convected further to the bottom of the pile. Thus, a strong segregation effect results in a pile with the small grains being found at the top of the pile and the large grains being found at the bottom of the pile. This effect has been observed in high-speed photography experiments to be important for segregation and stratification of granular mixtures [8] in well-developed flows of granular mixtures.

In this paper, we study analytically the segregation of granular mixtures poured in two-dimensional sandpiles due percolation in the rolling phase by using a suitable modification to the equations of motion for the rolling species interacting with the static grains of the pile. We calculate the corrections to the steady-state profiles due to the percolation effect and compare with theoretical predictions without percolation effects and with experimental observations.

Steady state solutions for the granular flows of grains in two-dimensional geometries have been calculated in [16] for the case of a single species flows in thin rotating drums, in [17] for the case of two-species differing in angle of repose and flowing in two-dimensional silos, and in [9,11] for thin flows of grains in two-dimensional silos with two species of grains differing in size and shape. When the grains differ only in angle of repose, the solution shows the segregation of the mixture and the concentrations decay slowly as a power-law of the position in the cell [17]. When there is a large difference in size and shape the solution shows a strong segregation of the mixture and the profiles decay exponentially fast [9,11]. The segregation solution is only valid for mixtures of large-rounded grains and small rough grains. Otherwise, the steady state solution is unstable, leading to the stratification of the mixture.

Here, we treat the case when the difference in size of the species is not too small (*i.e.* $d_2/d_1 > 1.4$) but, as opposed to [11], we focus on the thick flow regime. In this regime the thickness of the rolling phase is expected to be large, and it is possible to observe the percolation effect (the segregation in the rolling phase) [8]. The steady-state solution we find shows the complete segregation of the mixture with the large grains being found at the bottom of the pile, and we compare this solution with the solution corresponding to the case of thin flows when the percolation effect is absent.

The paper is organized as follows: in Section 2 we present the theoretical formalism for surface flows of granular mixtures. In Section 3 we calculate the steady-state solution of the problem, discuss our results, and compare to the theoretical predictions obtained without the percolation effect.

2 Surface flow of granular mixtures

The theoretical study of surface flows of granular materials was triggered by the works of Bouchaud, Cates, Prakash, and Edwards (BCRE) [18] and Mehta *et al.* [19]. In a recent theoretical study for the case of a single-species sandpile BCRE [18] proposed two coupled variables to describe the dynamics of two-dimensional sandpile surfaces: the local angle of the sandpile $\theta(x,t)$ (or alternatively the height of the sandpile $h(x,t)$) which describes the static phase (*i.e.*, the grains which belong to the pile), and the local thickness of the layer of rolling grains $R(x,t)$ to describe the rolling phase (*i.e.*, the grains that are not part of the pile but roll downwards on top of the static phase). BCRE also proposed a set of convective-diffusion equation for the rolling grains, which was later simplified by de Gennes [16].

Recently, Boutreux and de Gennes (BdG) [17] have extended the BCRE formalism to the case of two species. This formalism considers the two local "equivalent thicknesses" of the species in the rolling phase $R_\alpha(x,t)$ (*i.e.* the total thickness of the rolling phase multiplied by the local volume fraction of the α grains in the rolling phase at position x), with $\alpha = 1,2$ respectively for small and large grains. The total thickness of the rolling phase is defined as

$$R(x,t) \equiv R_1(x,t) + R_2(x,t). \tag{1}$$

The static phase is described by the height of the sandpile $h(x,t)$, and the volume fraction of static grains $\phi_\alpha(x,t)$ of type α at the surface of the pile. We consider a silo or cell of lateral size L and the pouring point is assumed to be at $x = 0$. For notational convenience we do not consider the difference between the angle and the tangent of the angle, *i.e.* $\theta(x,t) \equiv -\partial h/\partial x$. The equations of motion for the rolling species are [17]

$$\frac{\partial R_\alpha(x,t)}{\partial t} = -v_\alpha \frac{\partial R_\alpha}{\partial x} + \Gamma_\alpha, \tag{2a}$$

and the equation for $h(x,t)$ follows by conservation

$$\frac{\partial h(x,t)}{\partial t} = -\Gamma_1 - \Gamma_2. \tag{2b}$$

Here v_α is the downhill convection velocity of species α along x, assumed to be constant in space and in time. The interaction term Γ_α takes into account the conversion of static grains into rolling grains and *vice versa*, and it is defined as

$$\Gamma_1 \equiv a_1(\theta)\phi_1 R_1 - b_1(\theta) R_1 \tag{3a}$$

$$\Gamma_2 \equiv a_2(\theta)\phi_2 R_2 e^{-\lambda R_1/R} - b_2(\theta) R_2 e^{-\lambda R_1/R}. \tag{3b}$$

This definition involves a set of a priori unknown collision functions contributing to the rate processes: $a_\alpha(\theta)$ is the contribution due to an amplification process, (*i.e.*, when a static grain of type α is converted into a rolling grain due to a collision by a rolling grain of type α), $b_\alpha(\theta)$ is the contribution due to capture of a rolling grain of type α, (*i.e.*, when a rolling grain of type α is converted into a static grain). Cross-amplification processes, (*i.e.*, the amplification of a static grain of type β due to a collision by a rolling grain of type α) do not contribute to the exchange of grains since these kind of processes are inhibited by the percolation effect. Therefore, we do not consider them in the definition of the interaction term.

The interaction between rolling and static grains is assumed to be proportional to the number of interacting rolling grain, then it is proportional to $R_\alpha(x,t)$. This is clearly valid for thin flows. However, in the case of thick flows treated here, this approximation might still be valid [10] since the interaction might be proportional to the pressure exerted by the fluid phase which in turn is proportional to thickness of the rolling phase [20].

Due to percolation effect the interaction with the bulk of the large rolling grains $R_2(x,t)$ is greatly reduced as long as there are small rolling grains $R_1(x,t)$ interacting with the surface. Therefore, to take into account the percolation effect, we replace $R_2(x,t)$ in the definition of the interaction term by $R_2(x,t)\exp[-\lambda R_1(x,t)/R(x,t)]$. The exponential factor multiplying $R_2(x,t)$ mimics the fact that the interaction of large grains is screened by the presence of small grains so that R_2 interact with the grain at the surface of the static phase only when $R_1(x,t) \ll R(x,t)/\lambda$. The dimensionless parameter $\lambda > 0$ measures the degree of percolation, with $\lambda = 0$ corresponding to the no percolation case treated in [9].

The concentrations of static grains at the surface of the pile $\phi_\alpha(x,t)$ are given by

$$\phi_\alpha(x,t)\frac{\partial h}{\partial t} = -\Gamma_\alpha, \tag{4a}$$

and

$$\phi_1 + \phi_2 = 1. \tag{4b}$$

We use the following definitions of the collision functions [9]:

$$\begin{aligned} a_\alpha(\theta) &\equiv \gamma_{\alpha\alpha}\ \Pi[\theta(x,t) - \theta_\alpha(\phi_\beta)] \\ b_\alpha(\theta) &\equiv \gamma_{\alpha\alpha}\ \Pi[\theta_\alpha(\phi_\beta) - \theta(x,t)] \end{aligned} \tag{5a}$$

where

$$\Pi[x] \equiv \begin{cases} 0 & \text{if } \ x < 0 \\ x & \text{if } \ x \geq 0 \end{cases}. \tag{5b}$$

Here, the rates $\gamma_{\alpha\alpha} > 0$ has dimension of inverse time, and $v_\alpha/\gamma_{\alpha\alpha} \sim d_\alpha$. The generalized angle of repose $\theta_\alpha(\phi_\beta)$ of a α type of rolling grain is a continuous function of the composition of the surface ϕ_β [9], and we also define $\theta_{\alpha\beta}$ as $\theta_\alpha(\phi_\beta)$ for $\phi_\beta = 1$

$$\begin{aligned} \theta_1(\phi_2) &= m\phi_2 + \theta_{11} \\ \theta_2(\phi_2) &= m\phi_2 + \theta_{21} = -m\phi_1 + \theta_{22}, \end{aligned} \tag{6}$$

where $m \equiv \theta_{12} - \theta_{11} = \theta_{22} - \theta_{21}$. We assume the difference $\psi \equiv \theta_1(\phi_2) - \theta_2(\phi_2)$ to be independent of the concentration ϕ_2. We notice that for mixtures of grains with different shapes we have $\theta_{11} \neq \theta_{22}$ (here θ_{11} is the angle of the pure small species and θ_{22} denote the angle of repose of the pure large species), and different sizes $\theta_{12} \neq \theta_{21}$. In the following we treat only the case when the type 1 small grains are the roughest (*i.e.* $\theta_{22} < \theta_{11}$, since the angle of repose depends on the surface properties of the grains and it is larger for the roughest grains) or when the grains differ only in size (*i.e.*, $\theta_{22} = \theta_{11}$). The case when the large grains are the roughest ($\theta_{11} < \theta_{22}$) leads to stratification of the mixture and it is treated in [9–11].

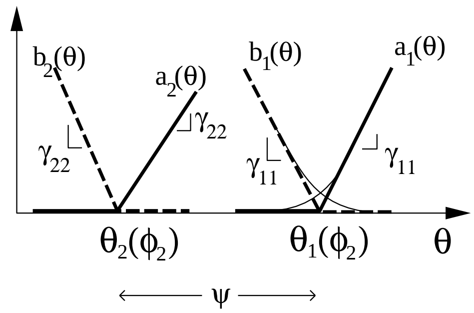

Fig. 2. Plot of capture $b_\alpha(\theta)$ and amplification $a_\alpha(\theta)$ functions. These functions are expected to be continuous in a region near the angle of repose (as shown by the thin curves). However, when the the ratio between the size of the grains is $d_2/d_1 > 1.4$ then $\psi = \theta_1(\phi_2) - \theta_2(\phi_2)$ is large enough so we can approximate these functions by the forms shown in this figure.

The collision functions a_α and b_α are expected to be continuous in the region of interest near the angle of repose (as shown, for instance, by the thin curves in Fig. 2). However, when the ratio between the size of the grains is large enough, the angular difference $\psi = \theta_1(\phi_2) - \theta_2(\phi_2)$ is expected to be large, and in this case, we can approximate the functions by the forms defined in (5). When d_2/d_1 is not close to one (*i.e.* $d_2/d_1 > 1.4$ according to experiments) strong segregation effects act in the system. In the case of thin flows treated in [9] (*i.e.* for $R(x,t) \lesssim 3 * d_2$) there is no percolation in the rolling phase and all grains (large and small) interact with the surface, but the large grains are not being captured because of the large difference in size. Thus the small grains are the only ones to be effectively interacting with the sandpile surface resulting in a strong segregation effect. If the flux of grains is large enough (*i.e.*, for $R(x,t) \gtrsim 3 * d_2$, the case treated here), segregation occurs in the rolling phase due to the percolation effect so that the large grains migrate to the top of the rolling phase and the small grains migrate to the bottom of the rolling phase [8,10,14,15]. Then small grains are the only ones interacting with the sandpile surface and again strong segregation is expected. This is confirmed by experiments done for $d_2/d_1 > 1.4$ showing sharp segregation patterns when the large grains are smoother than the small grains ($\delta \equiv \theta_{22} - \theta_{11} < 0$), and stratification when $\delta > 0$.

When the size ratio is close to one ($d_2/d_1 < 1.4$), the angle ψ is expected to be small and the collision functions a_α and b_α can be linearized around the respective angles of repose $\theta_\alpha(\phi_2)$, since the region of interest is localized to a small region around the angles of repose of the species. In this case segregation effects are expected to be less important in comparison with the above case. Indeed, according to the experiments [12] mixtures of grains differing in size and shape with size ratio close to one ($d_2/d_1 < 1.4$) give rise only to segregation and the concentration show slow decay, *i.e.* the segregation pattern is not sharp as in the above case. The case of weak segregation is treated theoretically in [21], where it is shown that the concentration profiles show slow (algebraic or power-law type) decay as a function of the position in the cell.

3 Profiles in the steady-state

We now calculate the steady-state solution of the equations of motion for the two-species sandpile including the effect of percolation.

We consider the geometry of a silo of lateral size L. We assume that the difference $\psi = \theta_1(\phi_2) - \theta_2(\phi_2)$ is independent of the concentration ϕ_2, then $\psi = \theta_{11} - \theta_{21} = \theta_{12} - \theta_{22}$. We set $v_1 = v_2 \equiv v$, and $\gamma_{11} = \gamma_{22} \equiv \gamma$. We seek a solution where the profiles of the sandpile and of the rolling grains do not change in time. Since stratification is an oscillatory solution, stratification cannot be observed for the steady-state solution. We set

$$\frac{\partial h}{\partial t} = v\ \frac{R^0}{L}, \tag{7a}$$

and

$$\frac{\partial R_\alpha(x)}{\partial t} = 0, \tag{7b}$$

with the following boundary conditions:

$$\begin{aligned} R_\alpha(x=0) &= R_\alpha^0, \\ R_\alpha(x=L) &= 0, \end{aligned} \tag{8}$$

with $R^0 \equiv R_1^0 + R_2^0$.

The profile of the total rolling species $R(x) = R_1(x) + R_2(x)$ is

$$R(x) = \frac{R^0}{L}(L - x). \tag{9}$$

which is obtained from equations (2a, 2b, 7). The linear profile for the total number of rolling species was also found in the case of one type of species in [16]. In fact this result is independent of the type of interaction term used, and it is a consequence of the conservation of grains.

The equations for the rolling species are obtained from equations (2a, 4a, 7) as a function of the concentrations ϕ_α

$$v\frac{\partial R_\alpha(x)}{\partial x} = -(vR^0/L)\ \phi_\alpha(x). \tag{10}$$

Since the collisions functions are defined according to the value of the angle in comparison to the generalized angles of repose, we divide the calculations in two regions: region A, where $\theta_2(\phi_2) < \theta < \theta_1(\phi_2)$, and region B, where $\theta < \theta_2(\phi_2) < \theta_1(\phi_2)$.

Region A

If $\theta_2 < \theta < \theta_1$, we obtain

$$\phi_1(x) = 1, \quad \phi_2(x) = 0, \tag{11}$$

then we obtain the profile of the rolling species using equations (10, 11)

$$R_1(x) = R_1^0 - \frac{R^0}{L}x, \quad R_2(x) = R_2^0. \tag{12}$$

The profile of the sandpile is obtained from equations (2b, 3a, 7a)

$$\theta(x) - \theta_{11} = \frac{-v/\gamma}{LR_1^0/R^0 - x}\cdot \tag{13}$$

This solution is valid when $\theta(x) > \theta_1(\phi_2 = 1) = \theta_{21}$, or for $x < x_m$, where x_m is

$$x_m = \frac{R_1^0}{R^0}L - \frac{v}{\gamma\psi}\cdot \tag{14}$$

We notice that in this region the solution is the same as the one found for the case of thin flows without percolation ($\lambda = 0$) in [11]. This is because, even though the large grains may interact with the pile surface when percolation effects are absent, the small grains are captured before the large grains since the surface at the top of the pile is made of small grains, so that the large grains roll down very easily and they are not trapped (in other words, the angle of repose of the pure small grains θ_{11} is larger than the angle of repose of the large grains rolling on a surface of small grains θ_{21}). The profile of the small rolling grains decays linearly with the distance as in the case of the total rolling species equation (9), since essentially the small grains are the only ones effectively interacting with the surface, so that it is a single species problem. Next, we show that the percolation effect shows up in the region at the bottom of the pile causing corrections to the exponential profiles of the concentrations found in [11].

Region B

If $\theta < \theta_2 < \theta_1$, from equations (2b, 3a, 7a) we obtain

$$\begin{aligned} -\frac{vR^0}{\gamma L} &= [\theta(x) - \theta_1(\phi_2)]R_1(x) \\ &\quad + [\theta(x) - \theta_2(\phi_2)]R_2(x)e^{-\lambda R_1/R}, \end{aligned} \tag{15}$$

and therefore

$$\theta(x) - \theta_2(\phi_2) = \frac{-vR^0/(\gamma L) + \psi R_1(x)}{R^*(x)}, \tag{16}$$

where $R^*(x) = R_1(x) + R_2(x)e^{-\lambda R_1/R}$. Using (16) we obtain the concentration as a function of the rolling species

$$\phi_1(x) = \frac{R_1(x)}{R(x)}\left(1 + \frac{\gamma\psi L}{vR^0}R_2(x)e^{-\lambda R_1/R}\right). \tag{17}$$

We obtain the equations for the rolling species using equations (10, 17),

$$\frac{\partial R_1(x)}{\partial x} = -\frac{R_1(x)}{R^*(x)}\left(\frac{R^0}{L} + \frac{R_2(x)e^{-\lambda R_1/R}}{r}\right), \qquad (18)$$

where $r \equiv v/(\gamma\psi)$. We set

$$u(x) = \frac{R_1(x)}{R^*(x)}, \qquad (19)$$

and we obtain from (18)

$$u'(x) = -u\left(-\frac{\partial R}{\partial x}\frac{1}{R^*(x)} + \frac{(1-u)e^{-\lambda u}}{r}\right) - \frac{\partial R^*(x)}{\partial x}\frac{1}{R^*(x)}\,. \qquad (20)$$

This equation cannot be solved in closed form, but to a good approximation we assume that $\partial R^*(x)/\partial x \approx \partial R(x)/\partial x$ (valid for small λ), and the equation is

$$u'(x) \approx -u(1-u)e^{-\lambda u}/r, \qquad (21)$$

which has the solution

$$\frac{R_1(x)}{R^*(x)} \approx \frac{1}{1 + Ce^{(x-x_m)/r}} \qquad (22)$$

where C is an integration constant obtained by considering the continuity at $x = x_m$. When there is no percolation effect ($\lambda = 0$) we recuperate the result found in [11], *i.e.* $R_1(x)/R(x) \sim \exp[-(x-x_m)/r]$ when $r \ll L$. Using equation (17) the concentration profiles can be obtained as a function of position.

To compare this solution with the case of no percolation effect, we obtain the exact numerical solution of equation (18). Figure 3 shows the solution we find for the rolling species with percolation in the rolling phase in comparison with the solution corresponding to no percolation effect $\lambda = 0$. We see that the percolation effect tends to increase the degree of segregation of the mixture in comparison with the results found in [9] without percolation as expected; the decay of the small rolling species when there is segregation in the rolling phase is faster than the exponential decay found when there is no segregation in the rolling phase.

The concentration of small grains decays very fast near the center of the pile as observed in experiments [8,12], with a region of mixing with a characteristic size of $r = v/\gamma\psi$. For typical experimental values corresponding to a system of $L = 30$ cm, $d_1 = 0.27$ mm, $d_2 = 0.8$ mm we have [8] $v \simeq 10$ cm/s, $\gamma \simeq 20$/s, $R^0 \simeq 0.5$ cm, $\tan\psi = \tan\theta_{11} - \tan\theta_{21} \simeq 0.2$ [22]. Therefore, the region of mixing in the center of the pile is of the order of $v/(\gamma\tan\psi) \simeq 2.5$ cm.

So far we have calculated the steady state solution of the problem. Numerical simulations [9–11] shows that this type of solution is only stable when the small grains are rougher than the large grains. When the large grains are the roughest, the solution turns unstable and the system displays the stratification pattern found in [7].

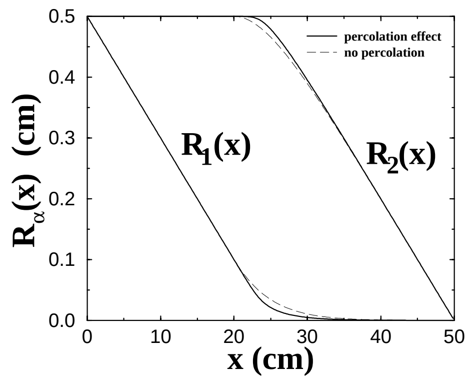

Fig. 3. Results of the profile of rolling species when the percolation effect is present. Notice that small species (type 1) are the only one interacting with the pile up to the center of the pile. Then the large grains start to be captured to, and the number of large rolling grains $R_2(x,t)$ decreases as well. For comparison we also plot in dashed line the solution corresponding to the case of thin flow where the percolation effect is not acting. We see that the segregation in the case of percolation is stronger than the segregation in the case of no percolation effect.

4 Discussion

In summary, we study analytically segregation in granular mixtures including the effect of segregation in the rolling phase, by introducing a suitable modification of the interaction term in the BdG equations. We find the corrections to the profiles of the steady-state of the filling of a silo. The profiles show strong segregation effects in agreement with experimental findings. The small corrections found here for the case of thick flows of grains with percolation in the rolling phase compared to the case of thin flows but with a large difference in size ratio treated in [11] suggests the applicability of the results found in [11] also to the case of thick flows.

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